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THEORY OF DEFORMATION OF ISOTROPIC HYPERELASTIC BODIES

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Deformation of isotropic hyperelastic bodies is considered. The solution of the problem of gravitational compression of a sphere is given as an example of application of the theory.

Key words: *isotropy, hyperelasticity, Jaumann stress rate, falling diagrams, gravitational compression.*

1. Rates of Squares of Basic Multiplicities of Extensions. In [1], in contrast to [2–4], it is proved that the strain energy density for isotropic hyperelastic bodies can be defined as a function of only two rather than three arguments: invariants of strain tensors. Components of the Jaumann stress-rate and strain rate tensors were used in [5] to study diagrams of stress dependences on strain. We find the relation between the strain rates and the rates of squares of basic multiplicities of extensions. We denote the Cartesian and curvilinear coordinates, radius vectors, basis vectors, and metric tensor of the curvilinear coordinate system as y^i, x^i, $\boldsymbol{R} = y^i\,\boldsymbol{k}_i$, $\boldsymbol{l}_i = \boldsymbol{R}_{,x^i} = y^n_{,x^i}\,\boldsymbol{k}_n$, and $g_{ij} = \boldsymbol{l}_i \cdot \boldsymbol{l}_j$ at the initial time $\tau = 0$ and as $\hat{y}^i$, $\hat{x}^i$, $\hat{\boldsymbol{R}} = \hat{y}^i\,\boldsymbol{k}_i$, $\hat{\boldsymbol{l}}_i = \hat{\boldsymbol{R}}_{,\hat{x}^i} = \hat{y}^n_{,\hat{x}^i}\,\boldsymbol{k}_n$, and $\hat{g}_{ij} = \hat{\boldsymbol{l}}_i \cdot \hat{\boldsymbol{l}}_j$ at the current time τ; $\boldsymbol{k}_i$ are the basis vectors of the Cartesian coordinate system; $\boldsymbol{u} = \hat{\boldsymbol{R}} - \boldsymbol{R} = u^i\,\boldsymbol{l}_i = \hat{u}^i\,\hat{\boldsymbol{l}}_i$ and $\boldsymbol{v} = \dot{\boldsymbol{u}} = \hat{v}_i\,\hat{\boldsymbol{l}}^i$ are the vectors of displacements and displacement rates; the dot indicates differentiation with respect to the time τ (in static problems, any other monotonically increasing parameter determining material deformation can be used instead of τ). We also use the basis vectors of the corresponding coordinate systems: $\hat{\boldsymbol{R}}_{,x^i} = \boldsymbol{l}_i + u^n_{,i}\,\boldsymbol{l}_n$ and $\boldsymbol{R}_{,\hat{x}^i} = \hat{\boldsymbol{l}}_i - \hat{u}^n_{;i}\,\hat{\boldsymbol{l}}_n$. The indices i, j, m, and n take the values 1, 2, and 3; summation from 1 to 3 is performed over repeated indices; the variables in the subscript after the comma indicate partial differentiation; the subscripts i after the comma or semicolon denote covariant differentiation with respect to x^i and $\hat{x}^i$, respectively ($\boldsymbol{u}_{,x^i} = u^n_{,i}\,\boldsymbol{l}_n$ and $\boldsymbol{u}_{,\hat{x}^i} = \hat{u}^n_{;i}\,\hat{\boldsymbol{l}}_n$).

Let the main axes of the Almansi strain tensor [1] $\hat{e} = \hat{e}_{ij}\hat{\boldsymbol{l}}^i\hat{\boldsymbol{l}}^j$ at the current time τ have directions of the basis vectors of the Cartesian coordinate system $\boldsymbol{k}_m$ and turn in the course of deformation with an angular velocity $\hat{\boldsymbol{\Omega}} = \hat{\Omega}^i\boldsymbol{k}_i$, though remaining mutually orthogonal. After a small time period, at the time $\tau + \Delta\tau$, the axes have the following directions with accuracy to $(\Delta\tau)^2$:

$$\hat{\boldsymbol{K}}^{(m)}_{\tau+\Delta\tau} = \boldsymbol{k}_m + \hat{\boldsymbol{\Omega}} \times \boldsymbol{k}_m\Delta\tau = \boldsymbol{k}_m + (\boldsymbol{k}_n\hat{\Omega}^l - \boldsymbol{k}_l\hat{\Omega}^n)\Delta\tau$$

[the indices m, n, and l take the values of an even permutation of 1, 2, and 3; the quantities with the index (m) refer to the axis with the direction $\boldsymbol{k}_m$ at the time τ; small terms of the order $(\Delta\tau)^2$ are neglected]. Elementary material fibers $\boldsymbol{k}_i\,d\hat{y}^i$ passing at the time τ along the main axes of the Almansi strain tensor $\hat{e}$ (no summation over i is performed) occupy the positions $(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{y}^i}\,d\hat{y}^i$ at the initial time, are mutually orthogonal, and pass along the main axes of the Green strain tensor $e = e_{ij}\boldsymbol{l}^i\boldsymbol{l}^j$. The ratios of the squared lengths of these fibers in the current and initial states are the squares of the main multiplicities of extensions $\varepsilon_i = |(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{y}^i}|^{-2}$. We consider the fibers $d\hat{L}^{(m)}_{\tau+\Delta\tau}\hat{\boldsymbol{K}}^{(m)}_{\tau+\Delta\tau} = (\hat{\boldsymbol{R}} + \boldsymbol{v}\Delta\tau)_{,\hat{x}^i}\,d\hat{x}^{i(m)}$ of length $d\hat{L}^{(m)}_{\tau+\Delta\tau}$ passing along the main axes at the time $\tau + \Delta\tau$. In the expansion of these vectors with respect to the basis $\hat{\boldsymbol{l}}_i$, we obtain the equalities

$$d\hat{x}^{i(m)} + (\boldsymbol{v}_{,\hat{x}^j} \cdot \hat{\boldsymbol{l}}^i)\Delta\tau d\hat{x}^{j(m)} = [\hat{x}^i_{,\hat{y}^m} + (\hat{x}^i_{,\hat{y}^n}\hat{\Omega}^l - \hat{x}^i_{,\hat{y}^l}\hat{\Omega}^n)\Delta\tau]\,d\hat{L}^{(m)}_{\tau+\Delta\tau}. \tag{1.1}$$

Rejecting terms containing $\Delta\tau$, we find $d\hat{x}^{i(m)} = \hat{x}^i_{,\hat{y}^m}\,d\hat{L}^{(m)}_{\tau+\Delta\tau}$. Substituting these values into the terms with $\Delta\tau$ in (1.1), we find the increments of coordinates with accuracy to $(\Delta\tau)^2$:

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$$d\hat{x}^{i(m)} = \{\hat{x}^i_{,\hat{y}^m} + [\hat{x}^i_{,\hat{y}^n}\hat{\Omega}^l - \hat{x}^i_{,\hat{y}^l}\hat{\Omega}^n - (\boldsymbol{v}_{,\hat{y}^m} \cdot \hat{\boldsymbol{l}}^i)]\Delta\tau\}\, d\hat{L}^{(m)}_{\tau+\Delta\tau}.$$

At the time τ, the considered fibers have the form $\hat{\boldsymbol{l}}_i\, d\hat{x}^{i(m)} = [\boldsymbol{k}_m + (\boldsymbol{k}_n\hat{\Omega}^l - \boldsymbol{k}_l\hat{\Omega}^n - \boldsymbol{v}_{,\hat{y}^m})\Delta\tau]\, d\hat{L}^{(m)}_{\tau+\Delta\tau}$; the form of the fibers at the initial time $\tau = 0$ is

$$(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{x}^i}\, d\hat{x}^{i(m)} = \{(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{y}^m} + [(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{y}^n}\hat{\Omega}^l - (\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{y}^l}\hat{\Omega}^n - (\boldsymbol{v}_{,\hat{y}^m} \cdot \boldsymbol{k}_j)(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{y}^j}]\Delta\tau\}\, d\hat{L}^{(m)}_{\tau+\Delta\tau}. \quad (1.2)$$

The ratios of the squares of their lengths specified at the time $\tau + \Delta\tau$ to the initial lengths $\varepsilon_{m(\tau+\Delta\tau)} = d\hat{L}^{(m)2}_{\tau+\Delta\tau}|(\hat{\boldsymbol{R}} - \boldsymbol{u})_{,\hat{x}^i}\, d\hat{x}^{i(m)}|^{-2}$ are the squares of the main multiplicities of extensions at the considered material point at the time $\tau + \Delta\tau$. Using the Cartesian components of the strain-rate tensor $\hat{\eta}_{ij} = [(\boldsymbol{v}_{,\hat{y}^i} \cdot \boldsymbol{k}_j) + (\boldsymbol{v}_{,\hat{y}^j} \cdot \boldsymbol{k}_i)]/2$, we obtain the expressions $\varepsilon_{m(\tau+\Delta\tau)} = \varepsilon_m(1 + 2\hat{\eta}_{mm}\Delta\tau)$. The rates of the squares of the main multiplicities of extensions are found through the limiting transition

$$\dot{\varepsilon}_m = \lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau}\,(\varepsilon_{m(\tau+\Delta\tau)} - \varepsilon_m) = 2\varepsilon_m\hat{\eta}_{mm}. \quad (1.3)$$

Thus, in the Cartesian coordinate system with the coordinate axes directed along the main axes of the Almansi strain tensor, the values of diagonal components of the strain-rate tensor are calculated by the formulas [6] $\hat{\eta}_{mm} = \dot{\varepsilon}_m/(2\varepsilon_m)$. The quantities $\dot{\varepsilon}_m$ are independent of the velocity of revolution of the main axes and are also the rates of the squares of multiplicities of extensions of material fibers passing along these axes at the time τ. Indeed, during the period from the time τ to the time $\tau + \Delta\tau$, the fibers $\boldsymbol{k}_m\, d\hat{y}^m$ (no summation over m is performed) pass to the positions $(\hat{\boldsymbol{R}} + \boldsymbol{v}\Delta\tau)_{,\hat{y}^m}\, d\hat{y}^m$. The ratios of the squared lengths of these fibers at the times $\tau + \Delta\tau$ and τ are equal to the ratios of their squared multiplicities of extensions at these times $|(\hat{\boldsymbol{R}} + \boldsymbol{v}\Delta\tau)_{,\hat{y}^m}|^2 = \varepsilon_{m(\tau+\Delta\tau)}\varepsilon_m^{-1} = 1 + 2\hat{\eta}_{mm}\Delta\tau$. Directing $\Delta\tau$ to zero, we find the same values of $\dot{\varepsilon}_m$ as in (1.3). From the condition of orthogonality of the vectors (1.2), we determine the velocity of revolution of the main axes $\hat{\Omega}^l = \omega^l + (\varepsilon_m + \varepsilon_n)(\varepsilon_m - \varepsilon_n)^{-1}\hat{\eta}_{mn}$ ($\boldsymbol{\omega} = \omega^i\boldsymbol{k}_i$ is the velocity of revolution of the neighborhood of the material point as an absolutely solid integer; $\omega^l = [(\boldsymbol{v}_{,\hat{y}^m} \cdot \boldsymbol{k}_n) - (\boldsymbol{v}_{,\hat{y}^n} \cdot \boldsymbol{k}_m)]/2$).

2. Stress Rates. Let the main axes of the Cauchy stress tensor $\hat{\sigma} = \hat{\sigma}^{ij}\,\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}_j$ at the time τ have the directions $\boldsymbol{k}_m$ and, remaining mutually orthogonal, turn with an angular velocity $\tilde{\boldsymbol{\Omega}} = \tilde{\Omega}^i\,\boldsymbol{k}_i$. Then, at the time $\tau + \Delta\tau$, with accuracy to $(\Delta\tau)^2$, the axes have the directions $\hat{\boldsymbol{N}}^{(m)}_{\tau+\Delta\tau} = \boldsymbol{k}_m + (\boldsymbol{k}_n\tilde{\Omega}^l - \boldsymbol{k}_l\tilde{\Omega}^n)\Delta\tau$ (the indices m, n, and l take the values of an even permutation of 1, 2, and 3). We consider material sites that have the normals $\hat{\boldsymbol{N}}^{(m)}_{\tau+\Delta\tau}$ and areas $d\hat{S}^{(m)}_{\tau+\Delta\tau}$ at the time $\tau + \Delta\tau$. Assuming that the normals to these material sites at the time τ are $\boldsymbol{k}_m$ in the first approximation, we find [5] $d\dot{\hat{S}}^{(m)} = (J^{-1}\dot{J} - \hat{\eta}_{mm})d\hat{S}^{(m)}$ and $\dot{\hat{\boldsymbol{N}}}^{(m)} = \hat{\eta}_{mm}\boldsymbol{k}_m - (\boldsymbol{v}_{,\hat{x}^i} \cdot \boldsymbol{k}_m)\hat{\boldsymbol{l}}^i$, where J is the ratio of volumes of material particles in the current and initial states (Jacobian of transformation of the initial Cartesian coordinates of material points to the current coordinates); $\dot{J} = J\,\hat{\eta}^i_i$. Using the formulas $d\hat{S}^{(m)} = d\hat{S}^{(m)}_{\tau+\Delta\tau} - d\dot{\hat{S}}^{(m)}\,\Delta\tau$ and $\hat{\boldsymbol{N}}^{(m)} = \hat{\boldsymbol{N}}^{(m)}_{\tau+\Delta\tau} - \dot{\hat{\boldsymbol{N}}}^{(m)}\Delta\tau$, we find the areas and normals to the sites at the time τ with accuracy to $(\Delta\tau)^2$:

$$\begin{gathered} d\hat{S}^{(m)} = [1 - (J^{-1}\dot{J} - \hat{\eta}_{mm})\Delta\tau]\, d\hat{S}^{(m)}_{\tau+\Delta\tau}, \\ \hat{\boldsymbol{N}}^{(m)} = (1 - \hat{\eta}_{mm}\Delta\tau)\boldsymbol{k}_m + [\boldsymbol{k}_n\tilde{\Omega}^l - \boldsymbol{k}_l\tilde{\Omega}^n + (\boldsymbol{v}_{,\hat{x}^i} \cdot \boldsymbol{k}_m)\hat{\boldsymbol{l}}^i]\Delta\tau. \end{gathered} \quad (2.1)$$

The areas and normals $dS^{(m)}$ and $\boldsymbol{N}^{(m)} = N^{(m)}_i\boldsymbol{l}^i$ at the initial time $\tau = 0$ are found from the equality [5] $\boldsymbol{N}^{(m)}J\, dS^{(m)} = (\hat{\boldsymbol{N}}^{(m)} \cdot \hat{\boldsymbol{R}}_{,x^i})\boldsymbol{l}^i\, d\hat{S}^{(m)}$, where

$$N^{(m)}_i = \frac{d\hat{S}^{(m)}}{JdS^{(m)}}\,\{(1 - \hat{\eta}_{mm}\Delta\tau)\hat{y}^m_{,x^i} + [\tilde{\Omega}^l\hat{y}^n_{,x^i} - \tilde{\Omega}^n\hat{y}^l_{,x^i} + (\boldsymbol{v}_{,x^i} \cdot \boldsymbol{k}_m)]\Delta\tau\}. \quad (2.2)$$

We determine the forces acting per unit area of the considered material sites at the time $\tau + \Delta\tau$:

$$\hat{\boldsymbol{q}}^{(m)}_{\tau+\Delta\tau} = (\sigma^{ij} + \dot{\sigma}^{ij}\Delta\tau)(\hat{\boldsymbol{R}} + \boldsymbol{v}\Delta\tau)_{,x^i}N^{(m)}_j\, dS^{(m)}(d\hat{S}^{(m)}_{\tau+\Delta\tau})^{-1}.$$

We pass from the second symmetric Piola–Kirchhoff stress tensor $\sigma = \sigma^{ij}\,\boldsymbol{l}_i\boldsymbol{l}_j$ to the Cauchy stress tensor $\hat{\sigma}^{mn} = J^{-1}\sigma^{ij}\hat{x}^m_{,x^i}\hat{x}^n_{,x^j}$ and introduce the tensor $\hat{s} = \hat{s}^{mn}\,\hat{\boldsymbol{l}}_m\hat{\boldsymbol{l}}_n$ ($\hat{s}^{mn} = J^{-1}\dot{\sigma}^{ij}\,\hat{x}^m_{,x^i}\hat{x}^n_{,x^j}$). Using (2.1) and (2.2) and neglecting small terms of the order of $(\Delta\tau)^2$, we obtain the expressions

$$\hat{\boldsymbol{q}}^{(m)}_{\tau+\Delta\tau} = \{(1 - J^{-1}\dot{J}\Delta\tau)\hat{y}^m_{,\hat{x}^i} + [\tilde{\Omega}^l\hat{y}^n_{,\hat{x}^i} - \tilde{\Omega}^n\hat{y}^l_{,\hat{x}^i} + (\boldsymbol{v}_{,\hat{x}^i} \cdot \boldsymbol{k}_m)]\Delta\tau\}\hat{\sigma}^{ij}\,\hat{\boldsymbol{l}}_j + (\hat{\sigma}^{ij}\boldsymbol{v}_{,\hat{x}^i} + \hat{s}^{ij}\,\hat{\boldsymbol{l}}_i)\,\hat{y}^m_{,\hat{x}^j}\Delta\tau$$

(the indices m, n, and l take the values of an even permutation of 1, 2, and 3). In the Cartesian coordinate system with the coordinate axes directed along the main axes of the tensor $\hat{\sigma}$, the expressions take the form

$$\hat{\boldsymbol{q}}^{(m)}_{\tau+\Delta\tau} = (1 - J^{-1}\dot{J}\Delta\tau)\hat{\sigma}_m \boldsymbol{k}_m + [\tilde{\Omega}^l \hat{\sigma}_n \boldsymbol{k}_n - \tilde{\Omega}^n \hat{\sigma}_l \boldsymbol{k}_l + (\boldsymbol{v}_{,\hat{y}^i} \cdot \boldsymbol{k}_m)\hat{\sigma}_i \boldsymbol{k}_i + \hat{\sigma}_m \boldsymbol{v}_{,\hat{y}^m} + \hat{s}^{mi}\, \boldsymbol{k}_i]\Delta\tau, \tag{2.3}$$

where $\hat{s} = \hat{s}^{ij}\, \boldsymbol{k}_i \boldsymbol{k}_j$.

The projections of vectors (2.3) onto the normals and tangents to the sites where they are determined, $Q^{ij}_{\tau+\Delta\tau} = \hat{\boldsymbol{q}}^{(i)}_{\tau+\Delta\tau} \cdot \hat{\boldsymbol{N}}^{(j)}_{\tau+\Delta\tau}$ are the stresses acting on these sites and satisfying the conditions $Q^{ij}_{\tau+\Delta\tau} = Q^{ji}_{\tau+\Delta\tau}$. The normal stresses acquire the values of the main components of stresses at the time $\tau + \Delta\tau$: $Q^{ii}_{\tau+\Delta\tau} = \hat{\sigma}_{i(\tau+\Delta\tau)} = \hat{\sigma}_i + \hat{\Sigma}^{ii}\Delta\tau$ (no summation over i is performed). Under the condition of zero tangent stresses $Q^{mn}_{\tau+\Delta\tau} = [\hat{\Sigma}^{mn} + (\hat{\sigma}_n - \hat{\sigma}_m)(\tilde{\Omega}^l - \omega^l)]\Delta\tau = 0$ (the indices m, n, and l take the values of an even permutation of 1, 2, and 3), we find the velocity of revolution of the main axes $\tilde{\Omega}^l = \omega^l + (\hat{\sigma}_m - \hat{\sigma}_n)^{-1}\hat{\Sigma}^{mn}$. Here, $\hat{\Sigma}^{ii} = \hat{s}^{ii} + \hat{\sigma}_i(2\hat{\eta}_{ii} - J^{-1}\dot{J})$, $\hat{\Sigma}^{mn} = \hat{s}^{mn} + (\hat{\sigma}_m + \hat{\sigma}_n)\hat{\eta}_{mn}$ are the components of the Jaumann stress-rate tensor $\hat{\Sigma} = \hat{\Sigma}^{ij}\boldsymbol{k}_i\boldsymbol{k}_j$ [7] in the Cartesian coordinate system with the coordinate axes directed along the main axes $\hat{\sigma}$. The diagonal components $\hat{\Sigma}^{ii}$ take the values of rates of the main stress components:

$$\hat{\Sigma}^{ii} = \dot{\hat{\sigma}}_i = \lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau}\,(\hat{\sigma}_{i(\tau+\Delta\tau)} - \hat{\sigma}_i). \tag{2.4}$$

In an isotropic hyperelastic body, the main stress components are determined from the equations [5]

$$\hat{\sigma}_i = \hat{\mu}\,\varepsilon_i\,(\varepsilon_i - \hat{\chi}) + p, \tag{2.5}$$

where $\hat{\mu} = \beta I_1^{-2}J^{-1}$; $\beta = \Psi_{,\Upsilon}$; $p = \Psi_{,J} = (\hat{\sigma}_m + \hat{\sigma}_n + \hat{\sigma}_l)/3$ is the hydrostatic pressure; $\Psi = \Psi(\Upsilon, J)$ is the strain energy density; $\hat{\chi} = 2I_1(\Upsilon + 1/3)$;

$$J = (\varepsilon_m\varepsilon_n\varepsilon_l)^{1/2}, \qquad \Upsilon = \frac{\varepsilon_m^2 + \varepsilon_n^2 + \varepsilon_l^2}{(\varepsilon_m + \varepsilon_n + \varepsilon_l)^2} - \frac{1}{3}, \qquad I_1 = \frac{1}{2}\,(\varepsilon_m + \varepsilon_n + \varepsilon_l)$$

(the indices m, n, and l take the values of an even permutation of 1, 2, and 3). From the formulas given above for the components of the Jaumann stress-rate tensor, we obtain the following expressions for isotropic hyperelastic bodies:

$$\hat{\Sigma}^{ii} = [(\dot{J}\beta_{,J} + \dot{\Upsilon}\beta_{,\Upsilon})\beta^{-1} - 2I_1^{-1}\dot{I}_1 - J^{-1}\dot{J} + 2\hat{\eta}_{ii}](\hat{\sigma}_i - p)$$
$$+ \hat{\mu}\varepsilon_i(2\varepsilon_i\hat{\eta}_{ii} - \dot{\hat{\chi}}) + \dot{J}p_{,J} + \dot{\Upsilon}p_{,\Upsilon}. \tag{2.6}$$

Substituting $\hat{\eta}_{ii} = \dot{\varepsilon}_i/(2\varepsilon_i)$ into (2.6), we obtain the values $\hat{\Sigma}^{ii} = \dot{\hat{\sigma}}_i$, which are also obtained by differentiation of $\hat{\sigma}_i$ in (2.5) with respect to τ. The conditions of falling diagrams of stresses versus strains $\hat{\Sigma}^{ii}\hat{\eta}_{ii} < 0$ used in [5] (no summation over i is performed) can now be represented in the form $\dot{\hat{\sigma}}_i\dot{\varepsilon}_i < 0$ [8–12]. Therefore, the diagram of the dependence of $\hat{\sigma}_i$ on ε_i is assumed to be falling if the extension of the fiber passing along the considered main axis is accompanied by a decrease in stress acting in this fiber and if fiber shrinkage is accompanied by an increase in stress. Note, the matrix of coefficients at $\dot{\varepsilon}_i$ in (2.6) is not symmetric, in contrast to the symmetric matrix in equations for the rates of the main components of the Piola–Kirchhoff stress tensor σ: $\sigma_i = 2\Psi_{,\varepsilon_i} = J\varepsilon_i^{-1}\hat{\sigma}_i$ and $\dot{\sigma}_i = 2\Psi_{,\varepsilon_i\varepsilon_m}\dot{\varepsilon}_m$.

The extradiagonal components of the Jaumann stress-rate and strain-rate tensors $\hat{\Sigma}^{mn}$ and $\hat{\eta}_{mn}$ multiplied by $\Delta\tau$ are increments of shear stresses and shear strains on material sites where the main stress components act at the time τ, whereas tangential stresses and shear strains equal zero. Therefore, the resultant increments are also stresses and strains at the time $\tau + \Delta\tau$. In isotropic hyperelastic bodies, they are related as

$$\hat{\Sigma}^{mn} = B_l\hat{\eta}_{mn}, \qquad B_l = \frac{\hat{\mu}(\varepsilon_m + \varepsilon_n)}{\varepsilon_m + \varepsilon_n + \varepsilon_l}\,[2\varepsilon_m\varepsilon_n + (\varepsilon_m + \varepsilon_n)\varepsilon_l - \varepsilon_l^2]$$

(the indices m, n, and l take the values of an even permutation of 1, 2, and 3).

In the initial nondeformed state of the material, we have $B_l = 2\mu_0 > 0$, where μ_0 is the shear modulus. For the shear stresses and shear strains caused by the rates $\hat{\Sigma}^{mn}$ and $\hat{\eta}_{mn}$ to have an identical direction, the coefficients B_l should be positive. The conditions $B_l > 0$ and $\beta > 0$ yield the inequalities [5]

$$\varepsilon_l < (\varepsilon_m + \varepsilon_n)/2 + [(\varepsilon_m + \varepsilon_n)^2/4 + 2\varepsilon_m\varepsilon_n]^{1/2}, \tag{2.7}$$

independent of the form of the function $\Psi(\Upsilon, J)$. According to (2.7), the inequalities $\hat{\sigma}_m < \hat{\sigma}_n < \hat{\sigma}_l$ should be satisfied for the main stress components if $\varepsilon_m < \varepsilon_n < \varepsilon_l$. Note, by virtue of (2.5) and expressions for $\hat{\Sigma}^{mn}$, the found velocities of revolution of the main axes of the tensors $\hat{\sigma}$ and $\hat{e}$ coincide: $\hat{\boldsymbol{\Omega}} = \tilde{\boldsymbol{\Omega}}$.

3. Geometry of the Region Determined by Inequalities (2.7). In a three-dimensional space with the Cartesian coordinates $\varepsilon_i \geqslant 0$, inequalities (2.7) determine a cone whose apex is located in the origin $\varepsilon_i = 0$.

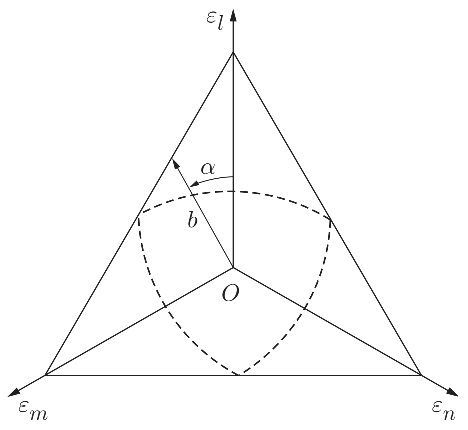

Fig. 1

We find the cross section of this cone by the deviatoric plane $\varepsilon_m + \varepsilon_n + \varepsilon_l = 3a$ (the indices m, n, and l take the values of an even permutation of 1, 2, and 3), which is located at a distance $a\sqrt{3}$ from the origin. The points of the deviatoric plane whose coordinates satisfy the inequality $\varepsilon_i \geqslant 0$ are located inside an equilateral triangle (Fig. 1) with apices on the coordinate axes and sides of length $3a\sqrt{2}$ located in the coordinate planes. In what follows, these sides are called the lines $\varepsilon_i = 0$. We introduce a polar coordinate system (b, α) with the origin at the center of the triangle (b is the radius and α is the polar angle). In this coordinate system, $\varepsilon_l = a(1 + \upsilon\cos\alpha)$, $\varepsilon_m = a[1 - \upsilon\cos(\alpha + \pi/3)]$, $\varepsilon_n = a[1 - \upsilon\cos(\alpha - \pi/3)]$, where $0 \leqslant \upsilon = (b/a)\sqrt{2/3} \leqslant 2$; $0 \leqslant \Upsilon = \upsilon^2/6 \leqslant 2/3$; $b = [(\varepsilon_m - a)^2 + (\varepsilon_n - a)^2 + (\varepsilon_l - a)^2]^{1/2}$; for $1 < \upsilon \leqslant 2$, the angles vary in the intervals $-\pi + \alpha_* \leqslant \alpha \leqslant -\pi/3 - \alpha_*$, $-\pi/3 + \alpha_* \leqslant \alpha \leqslant \pi/3 - \alpha_*$, $\pi/3 + \alpha_* \leqslant \alpha \leqslant \pi - \alpha_*$ [$\alpha_* = \arccos(1/\upsilon)$]. With the use of these expressions, the equality in (2.7) $\varepsilon_l = (\varepsilon_m + \varepsilon_n)/2 + [(\varepsilon_m + \varepsilon_n)^2/4 + 2\varepsilon_m\varepsilon_n]^{1/2}$ transforms to $\cos\alpha = 1/\upsilon - \upsilon/2$ and defines, in the Cartesian coordinates $X = b\cos\alpha$, $Y = b\sin\alpha$, the arc of the circumference $(X + a\sqrt{3/2})^2 + Y^2 = 9a^2/2$ connecting the middle points of the lines $\varepsilon_m = 0$ and $\varepsilon_n = 0$, whose center ($X = -a\sqrt{3/2}$, $Y = 0$) is located at the mid-point of the line $\varepsilon_l = 0$. Thus, in the deviatoric plane, inequalities (2.7) determine a curvilinear triangle whose sides are arcs of circumferences (Fig. 1). For the inequalities $B_l > 0$ to be satisfied, the values of the squares of the main multiplicities of extensions ε_i should be located in the region inside the conical surface with generatrices crossing deviatoric planes on the sides of curvilinear triangles.

4. Gravitational Compression of a Sphere. We determine spherically symmetric equilibrium states of the sphere. We denote the initial and current radial coordinates of the material points as r and $\hat{r}$ [$\hat{r} = \hat{r}(r)$ and $0 \leqslant r \leqslant R$]; $\varepsilon_1 = (\hat{r}_{,r})^2$ and $\varepsilon_2 = (\hat{r}/r)^2$ are the squares of the main multiplicities of extensions; $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are the radial and circumferential stresses; $J = \varepsilon_2\varepsilon_1^{1/2}$; the inequalities $\hat{r} \geqslant 0$ and $\hat{r}_{,r} > 0$ are satisfied. The density of the sphere material changes from the initial constant value ρ to $\hat{\rho} = \rho J^{-1}$ in the compressed state. We obtain the equilibrium equation

$$(\hat{r}^2\hat{\sigma}_1)_{,r} - \hat{\sigma}_2(\hat{r}^2)_{,r} + qr^2 = 0. \tag{4.1}$$

At each material point, the action of gravitational forces from the side of the entire sphere reduces to the force of attraction $q = -G\rho M\hat{r}^{-2}$ determined per unit of the initial volume of the material and directed toward the sphere center [G is the gravitational constant and $M = (4/3)\pi r^3\rho$]. Zero stress $\hat{\sigma}_1 = 0$ is set on the surface $r = R$; in the center of the sphere, we have $\varepsilon_1 = \varepsilon_2$ for $r = 0$.

The material of the sphere is assumed to be isotropic and hyperelastic, with the constitutive function [5] $\Psi = \beta\Upsilon + 0.5KJ^{-1}(J-1)^2$ continuously transforming, as the strain tends to zero, to the constitutive function of Hooke's law, with the same two constants of the material as those in Hooke's law: $\beta = 9\mu_0/4$, $\mu_0 = E_0/(2(1+\nu))$, and $K = E_0/(3(1-2\nu))$ (E_0 and ν are Young's modulus and Poisson's ratio). For $\hat{\sigma}_2 = \hat{\sigma}_3$ and $\varepsilon_2 = \varepsilon_3$, Eq. (2.5) yields

$$\hat{\sigma}_1 = \frac{8\beta\varepsilon_1^{1/2}(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + 2\varepsilon_2)^3} + p, \qquad \hat{\sigma}_2 = \frac{1}{2}(3p - \hat{\sigma}_1), \quad p = \frac{K}{2}\Big(1 - \frac{1}{J^2}\Big). \tag{4.2}$$

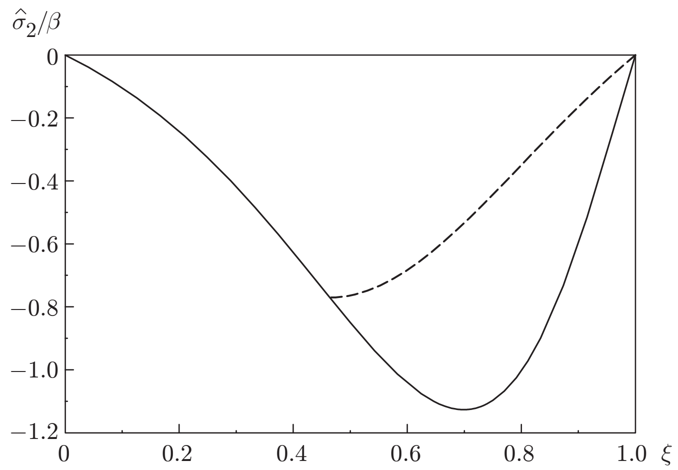

Fig. 2

According to (4.2), the sphere surface is characterized by the states described by the functions $\gamma = 16\xi(\xi^2 - 1)/(3 - \xi)^3$, $J = \gamma/c + [(\gamma/c)^2 + 1]^{1/2}$, $c = K/\beta = 8(1 + \nu)/(27(1 - 2\nu))$, $\varepsilon_1 = J^{2/3}((1 + \xi)/(1 - \xi))^{2/3}$, $\varepsilon_2 = J^{2/3}((1-\xi)/(1+\xi))^{1/3}$, $p = \beta\gamma/J$, $\hat{\sigma}_1 = 0$, and $\hat{\sigma}_2 = 3p/2$ depending on the parameter $0 \leqslant \xi = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2) \leqslant 1$ (in [5], ξ has the opposite sign). The absolute value of the stress $\hat{\sigma}_2$ increases and reaches a maximum (of the order of the shear modulus μ_0) at $\xi = \xi_* = (2\sqrt{7} + 1)/9$, after which $\hat{\sigma}_2 \to 0$ and $J \to 1$ as $\xi \to 1$ (solid curve in Fig. 2). (The curves in Figs. 2 and 3 were obtained for $\nu = 0.3$.) For $\xi > \xi_*$, the condition of a decreasing dependence of stress on strain is fulfilled: $\hat{\sigma}_{2,\xi}\, \varepsilon_{2,\xi} < 0$.

At the time $\xi = \xi_{**} = 2\sqrt{3} - 3 < \xi_*$, solutions of Eqs. (2.5) with deformation asymmetric relative to the radial beam branch off from this solution at $\varepsilon_2 = \varepsilon_3$; in these solutions, $\hat{\sigma}_1 = 0$, $\hat{\sigma}_2 = \hat{\sigma}_3 = 3\beta\gamma_1/(2J)$, $B_1 = 0$, $\gamma_1 = -4\zeta/(3 + \sqrt{1+\zeta})^2$, $J = \gamma_1/c + [(\gamma_1/c)^2 + 1]^{1/2}$, $\varepsilon_1 = \zeta_1(1 + \sqrt{1+\zeta}\,)$, $\varepsilon_2 = \zeta_1(1 - \sqrt{1 - \zeta/2}\,)$, $\varepsilon_3 = \zeta_1(1 + \sqrt{1 - \zeta/2}\,)$, $\zeta_1 = (\varepsilon_2 + \varepsilon_3)/2 = (2J^2/(\zeta(1 + \sqrt{1+\zeta}\,))^{1/3}$, the value of ζ decreases from 2 to 0. Asymmetric deformation occurs with decreasing stress $\hat{\sigma}_2$, which is lower than that in the solution with $\varepsilon_2 = \varepsilon_3$ (dashed curve in Fig. 2). The values of the parameter $\xi < \xi_{**}$ are set below for spherically symmetric equilibrium states of the sphere. The values of ξ_* and ξ_{**} are independent of material constants.

We substitute $q = -(4/3)\pi G\rho^2 r\varepsilon_2^{-1}$ and the expressions for $\hat{\sigma}_1$, $\hat{\sigma}_2$ from (4.2) into (4.1). Using the equation $\varepsilon_{2,r} = (2/r)(\sqrt{\varepsilon_1\varepsilon_2} - \varepsilon_2)$, we obtain the system of first-order differential equations in dimensionless variables

$$\varepsilon_{1,r'} + \frac{1}{f_1}\Big(\frac{f_2}{r'} - \frac{Ar'}{\varepsilon_2}\Big) = 0, \qquad \varepsilon_{2,r'} = \frac{1}{r'}\, f_3 \tag{4.3}$$

with respect to two sought functions ε_1 and ε_2, where

$$f_1 = \frac{4\varepsilon_2(11\varepsilon_1\varepsilon_2 - 3\varepsilon_1^2 - 2\varepsilon_2^2)}{\sqrt{\varepsilon_1}(\varepsilon_1 + 2\varepsilon_2)^4} + \frac{c}{2\varepsilon_1^2\varepsilon_2}, \qquad f_3 = 2(\sqrt{\varepsilon_1\varepsilon_2} - \varepsilon_2),$$

$$f_2 = \Big(\frac{8\sqrt{\varepsilon_1}\varepsilon_2(4\varepsilon_2 - 7\varepsilon_1)}{(\varepsilon_1 + 2\varepsilon_2)^4} + \frac{c}{\varepsilon_1\varepsilon_2^2}\Big) f_3 + \frac{24\varepsilon_1\sqrt{\varepsilon_2}(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + 2\varepsilon_2)^3},$$

with constants $c = 8(1+\nu)/(27(1-2\nu))$ and $A = 4G\pi\rho^2R^2/(3\beta)$ for $0 \leqslant r' = r/R \leqslant 1$. The solution of system (4.3) that satisfies the boundary conditions formulated above is calculated by the Runge–Kutta method as a solution of the problem with initial conditions set for $r' = 1$. To eliminate uncertainty in the sphere center, the calculations are performed only on the sector from $r' = 1$ to $r' = r'_\delta = 0.0001$. The value of ξ is set, and the values of ε_1 and ε_2 on the sphere surface are calculated using the above-given formulas. Solving system (4.3) with specified initial conditions from $r' = 1$ to r'_δ, we use an iterative procedure to find the value of A providing calculation of the solution up to the point r'_δ where the equality $\varepsilon_1 = \varepsilon_2$ is satisfied with sufficient accuracy. It should be noted that the value of A is to be determined rather accurately (10–12 digits after the point in the representation of A to satisfy the equality $\varepsilon_1 = \varepsilon_2$ at the point r'_δ with accuracy to 10^{-7}). As a result, we find the solution with the value of A for which the specified value of ξ is obtained.

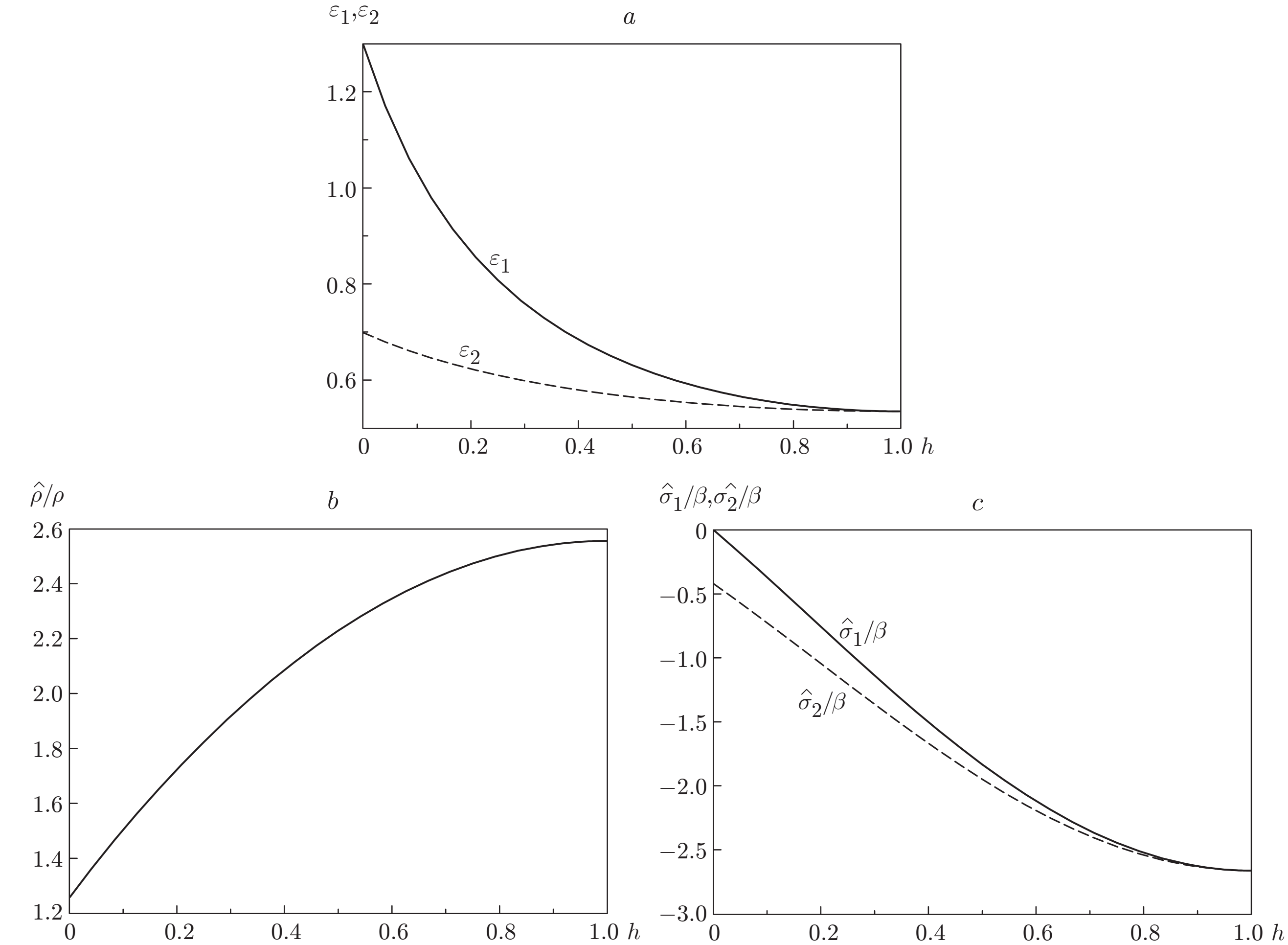

Fig. 3

Figure 3a–c shows the dependences of the squares of the main multiplicities of extensions ε_1 and ε_2, the ratios of densities $\hat{\rho}/\rho$, and the stresses $\hat{\sigma}_1/\beta$ and $\hat{\sigma}_2/\beta$ versus $h = 1 - r'$ for $\nu = 0.3$, $\xi = 0.3$, and $A = 2.2921373248$ [several last digits in the value of A can be different if another variable is taken as independent in (4.3), e.g., r'^2 instead of r'].

It should be noted that linearization of (4.1), (4.2) with the use of the equalities $\varepsilon_1 = 1+2e_1$ and $\varepsilon_2 = 1+2e_2$ under the condition of small strains e_1 and e_2, displacements $u = \hat{r} - r$, and their derivatives $u_{,r}$ leads to equations formulated in the linear theory of elasticity:

$$\sigma_{1,r} + \frac{2}{r}\,(\sigma_1 - \sigma_2) - A\,\frac{\beta r}{R^2} = 0, \qquad e_1 = u_{,r}, \qquad e_2 = \frac{u}{r},$$

$$\sigma_1 = \frac{8\beta}{9(1-2\nu)}\,[(1-\nu)e_1 + 2\nu e_2], \qquad \sigma_2 = \frac{8\beta}{9(1-2\nu)}\,(e_2 + \nu e_1).$$

From the solution of these equations with the boundary conditions $\sigma_1 = 0$ on the surface and $u = 0$ at the center of the sphere, we find the stresses

$$\sigma_1 = \frac{(3-\nu)\beta A}{10(1-\nu)}\Big(\frac{r^2}{R^2} - 1\Big), \qquad \sigma_2 = \frac{\beta A}{10(1-\nu)}\Big[(1+3\nu)\,\frac{r^2}{R^2} - (3-\nu)\Big],$$

which are lower than in the nonlinear theory; this fact is explained by the neglect of the changes in areas of material sites and distances between material points upon sphere compression. Thus, for $\nu = 0.3$ and $A \approx 2.29$, the pressure at the center of the sphere is $\sigma_1/\beta = \sigma_2/\beta \approx -0.88$, which is almost three times lower than that in Fig. 3c.

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